Journal of Geometry and Physics 35 (2000) 75–91

Veech groups and polygonal coverings

Pascal Hubert [a], Thomas A. Schmidt [b,*]

[a] *Institut de Mathématiques de Luminy, 163 av de Luminy, Case 907, 13288 Marseille Cedex 09, France*
[b] *Department of Mathematics, Oregon State University, Kidder Hall 368, Corvallis, OR 97331-4605, USA*

Received 9 October 1998; received in revised form 15 December 1999

Abstract

We discuss branch points of affine coverings and their effects on Veech groups. In particular, this allows us to show that even if one polygon tiles another, the respective Veech groups are not necessarily commensurable. We also show that there is no universal bound on the number of Teichmüller disks passing through the same point of Teichmüller space and having incommensurable lattice Veech groups. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 30F60; 58F17

Sub. Class: Differential geometry; Dynamical systems

Keywords: Flat surfaces; Teichmüller disks; Billiards

1. Introduction

One of the most accessible problems in physics and mathematics would seem to be that of the dynamics of a particle elastically reflected by the walls of a Euclidean polygon. This seemingly innocuous problem, generically called *billiards*, offers interest already in the case of rational polygons, where all angles are rational multiples of π. Here the phase space decomposes into invariant surfaces for the natural billiard flow.

The study of this billiard flow leads to quadratic differentials and Teichmüller space; for a survey on these matters, see [18]. This machinery has allowed such beautiful results as Masur's [17] density of periodic geodesics and the Kerkhoff–Masur–Smillie result [12], see also [1], on unique ergodicity of the flow (on each invariant surface). These results are in general true for almost every direction. Veech [21,22] gave explicit examples for which the billiard flow behaves like the linear flow on the torus: in each direction the flow is either periodic or it is uniquely ergodic.

* Corresponding author.
E-mail addresses: hubert@iml.univ-mrs.fr (P. Hubert), toms@math.orst.edu (T.A. Schmidt)

PII: S0393-0440(99)00080-7

The passage from billiards on a rational polygon to quadratic differentials is fairly natural. Speaking loosely, instead of following the trajectory of a particle as it reflects off a wall (i.e., edge), one can instead flip the polygon about the edge. This process defines a surface (in fact a Riemann surface, see below for references), and a quadratic differential on the surface. Actually, the quadratic differential that one finds is the square of a holomorphic 1-form. The surface being constructed from pieces of the plane has a locally flat structure with singularities. Veech had the insight to emphasize self-maps of such surfaces which are locally affine with respect to the flat structure. The matrices which are the derivatives of these affine maps form a group, the Veech group. Veech showed that whenever this group is of finite covolume, one has the above dichotomy for the directions of flow: the flow in each direction is either periodic or it is uniquely ergodic.

There are various examples known of Veech groups which are lattices [5,11,21–24]; i.e., of finite covolume. There are also various results, especially of Gutkin and Judge [6,7] and Kenyon and Smillie [11], indicating that these are rare.

Here we emphasize coverings of surfaces and pull-backs of forms in order to study relationships between Veech groups. The use of coverings in the study of Veech groups is already well established, especially by Vorobets [23] and by Gutkin and Judge. We use an algebraic approach for which the results and techniques of Aurell and Itzykson [3] are quite helpful.

Our main results are given as Theorems 1 and 2. We show that there is no universal upper bound on the number of non-commensurable lattice Veech groups that can be associated to a single Riemann surface. To this date, explicit examples only showed that this number could be as large as 2, see [5]. We also give both algebraic and geometric proofs showing that tiling of rational polygons by way of flips *does not* necessarily preserve commensurability of Veech groups of the related surfaces. We thank J. Smillie for pointing out to us that a remark in passing of Vorobets [23] already points to such counter-examples.

Our techniques are a clear use of ramification in coverings of the type treated by Gutkin–Judge and Vorobets. We combine this with fundamental work of Gutkin–Judge and Vorobets (see Theorem A), all eventually based upon the pioneering work of Veech (see Theorem B).

2. Background: translation structures and Veech groups

A surface is said to have a *translation structure* if it is equipped with a fixed atlas for which the transition functions are translations in $\mathbb{R}^2$. Translation structures are most naturally discussed in terms of holomorphic 1-forms.

There is a natural construction, apparently due to [10], of a surface with translation (also called flat or Euclidean) structure and conical singularities (see [20] for these terms) directly related to the billiard flow on a table formed by a rational polygon. Let us discuss the case of rational triangles.

Notation. Let $\mathcal{T}(p, q, r)$ be the rational Euclidean triangle whose angles are $p\pi/n, q\pi/n, r\pi/n$, where $n = p + q + r$ and 1=g.c.d.(p, q, r) (see Fig. 4).

By an unfolding process, one follows the straight line paths on $2(p+q+r)$ copies of the triangle. The free edges can be identified so as to obtain a surface with a translation structure. This construction allows one to pull-back $\mathrm{d}z$ from the plane and thus identify a 1-form on the surface, and thereby the holomorphic structure such that the form is holomorphic, see say [19]. The actual equation of the surface and the identification of the form were apparently not given until [3], see also [24]. The surface is the smooth Riemann surface associated to the equation $y^{p+q+r} = x^{p+r}(1-x)^{q+r}$, and the form there is $\mathrm{d}x/y$. This follows from an application of the Schwarz triangle function.

In the other direction, by integrating a holomorphic 1-form on a Riemann surface, one obtains charts of local coordinates which give the surface a translation structure with conical singularities at the zeros of the 1-form, again see [20]. Of course, the group $SL(2,\mathbb{R})$ has its usual action on $\mathbb{R}^2$; by composing this action with the local coordinate functions, one obtains an action of $SL(2,\mathbb{R})$ on the set of atlases on the surface. Each of the atlases so found also corresponds to a 1-form. Kerkhoff et al. [12] showed that the $SL(2,\mathbb{R})$ orbit of a holomorphic 1-form is the unit cotangent space to the so- called Teichmüller disk of the 1-form. See [5] for a very clear exposition of these matters.

2.1. Affine functions and Veech groups

Let us fix a form ω on a Riemann surface M and let $Z(\omega)$ denote the set of the zeros of ω. Let $M' := M \setminus Z(\omega)$. A diffeomorphism $f : M' \to M'$ which extends to a homeomorphism from M to itself is called *affine* with respect to the translation structure on M induced by ω if the derivative of f is constant in the charts of ω and is given by some fixed element $A \in SL(2,\mathbb{R})$. Note that this definition requires that the extension of f and its inverse send $Z(\omega)$ to itself (permutation of this set is allowed).

Away from zeros of ω, locally $f(z) = Az + c_i$, where the c_i depend only on the chart of z. The set of all such functions is called the affine group of ω, $\mathit{Aff}(\omega)$. The Veech group, $\Gamma(\omega)$, is the group of matrices representing the derivatives of the affine functions. In fact, Veech [21] shows that the object of main interest is this group taken up to projective equivalence; i.e., we need only consider the image of Γ in $PSL(2,\mathbb{R})$. In what follows, we will indeed simply write $\Gamma(\omega)$ for this corresponding subgroup of $PSL(2,\mathbb{R})$.

Each Teichmüller disk with its so-called Teichmüller metric is isometric to the hyperbolic plane, thus has $PSL(2,\mathbb{R})$ as full isometry group, see say [5]. Veech also showed that by way of the isometry $\Gamma(\omega)$ acts discontinuously on the disk of ω, $\mathcal{D}_\omega$. (In fact, Veech gave his results in the general setting of quadratic differentials. Now, the square of a 1-form is indeed a quadratic differential, and see say work of Kra [13], all quadratic differentials in the Teichmüller disk of a square quadratic differential are of this same type. By tacitly using the squares of our holomorphic 1-forms where necessary, we continue in our slightly simplified setting.)

As $\Gamma(\omega)$ is a subgroup of $PSL(2,\mathbb{R})$ which acts discontinuously on the hyperbolic plane, $\Gamma(\omega)$ is a Fuchsian group. The quotient of the disk $\mathcal{D}_\omega$ by $\Gamma(\omega)$ is a Riemann surface (with hyperbolic structure induced by the Teichmüller metric) inside the Riemann moduli space of M. A result of Masur implies that this surface cannot be compact. See [8,9] for some remarks on the possible relationship of this new surface to M.

In the setting of polygonal billiards, Veech demonstrated a direct relationship between a certain dynamical zeta function of the system and the hyperbolic metric of the new surface. He also calculated the Veech groups for a particular set of examples. In the ensuing decade, there have been some more examples discovered and various results obtained which indicate that the Veech group is rarely non-trivial.

2.2. Marking extra points

It is convenient to consider translation structures with some removable singularities marked. We introduce notation for this purpose.

Notation. Let $\mathcal{P}(\omega; \{p_1, \ldots, p_n\})$ denote the translation structure on a surface M given by the 1-form ω and having marked points $p_1, \ldots, p_n$ as well as the zeros of ω. Given $\mathcal{P}$ of this sort, let M'' be M (having the structure of ω) with both $Z(\omega)$ and the set of the p_i removed. The affine group, $\mathit{Aff}(\mathcal{P})$, for such a marked translation structure is the group of the affine diffeomorphisms which restrict so as to take M'' to itself. The Veech group, $\Gamma(\mathcal{P})$, is then the derivatives of these affine diffeomorphisms.

For a fixed surface M, and marked structures $\mathcal{P}$ and $\mathcal{Q}$, we write $\mathcal{P} \subset \mathcal{Q}$ if the marked structures have the same underlying 1-form, and the marked points of $\mathcal{P}$ are amongst those of $\mathcal{Q}$.

Recall that a subgroup of $PSL(2, \mathbb{R})$ is called a *lattice* if it acts discontinuously on the hyperbolic plane and the corresponding quotient is of finite volume. The following lemma was implied in a message from C. Judge.

Lemma 1. *Let $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P} \subset \mathcal{Q}$, be as above. Then both $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ are subgroups of $\Gamma(\omega)$. Furthermore, there is a finite index subgroup of $\Gamma(\mathcal{Q})$ which is contained in $\Gamma(\mathcal{P})$. If $\Gamma(\mathcal{Q})$ is a lattice, then so are $\Gamma(\mathcal{P})$ and $\Gamma(\omega)$.*

Proof. We show that $\mathit{Aff}(\mathcal{P}) \subset \mathit{Aff}(\omega)$. For this, it suffices that any affine diffeomorphism taking $M \setminus \{p_1, \ldots, p_n\}$ to itself can be extended so as to take the set of the p_i to itself. The diffeomorphism on M clearly acts as a permutation on $Z(\omega) \cup \{p_i\}$.

A diffeomorphism cannot remove any singularity of the translation structure which arises as an element of $Z(\omega)$. (Indeed, the order of a zero of the 1-form is invariant.) Thus, it in fact permutes the p_i. That is, the restriction to M' gives an element of $\mathit{Aff}(\omega)$. These arguments clearly show that $\mathit{Aff}(\mathcal{Q}) \subset \mathit{Aff}(\omega)$ as well.

Suppose that the marked points of $\mathcal{Q}$, in addition to those of $\mathcal{P}$, comprise the set $\{q_1, \ldots, q_m\}$. The previous paragraph shows that each $f \in \mathit{Aff}(\mathcal{Q})$ naturally gives a permutation $\bar{f}$ on $\{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_m\}$. This defines a group homomorphism from $\mathit{Aff}(\mathcal{Q})$ to the symmetric group $\mathit{Sym}(\{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_m\})$. The kernel of this homorphism is a finite index subgroup of $\mathit{Aff}(\mathcal{Q})$ which acts as the trivial permutation on $\{p_1, \ldots, p_n\}$.

Hence, this finite index subgroup of *Aff*($\mathcal{Q}$) is contained in *Aff*($\mathcal{P}$). Therefore, $\Gamma(\mathcal{Q})$ has a finite index subgroup contained in $\Gamma(\mathcal{P})$.

If a Fuchsian group has finite covolume, then so does any of its finite index subgroups. Also if any subgroup of a Fuchsian group has finite covolume, then the group itself does. Hence, we conclude that if $\Gamma(\mathcal{Q})$ is a lattice, then so are $\Gamma(\mathcal{P})$ and $\Gamma(\omega)$. □

2.3. Translation and affine coverings

We say that a map $f : M \to N$ gives a *translation covering* of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$ if the restriction $f : M'' \to N''$ is such that $\psi \circ f \circ \phi^{-1}$ are translations where ψ and ϕ are the (various appropriate choices of the) local coordinates for the atlases of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Note that a translation covering is in particular a holomorphic (ramified) covering of the corresponding Riemann surfaces.

Similarly, we say that a map f gives an *affine covering* of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$ if the restriction $f : M'' \to N''$ is such that the aforementioned compositions are of the form $Az + c_{i,j}$ where A is a fixed matrix in $SL(2, \mathbb{R})$, but the translation vectors $c_{i,j}$ may vary with the choice of charts. Note that an affine covering is in particular a quasi-conformal (ramified) covering of the corresponding Riemann surfaces.

Let B be any matrix in $SL(2, \mathbb{R})$. We define $(M, B \circ \mathcal{P})$ by replacing the coordinate functions of the translation structure of $(M, \mathcal{P})$ by their post-composition with B. Let f give an affine covering of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$. If A is the matrix of the derivative of f, then we define f^A to be the covering of $(N, \mathcal{Q})$ by $(M, A \circ \mathcal{P})$. Similarly, we define f_A to be the covering of $(N, A^{-1} \circ \mathcal{Q})$ by $(M, \mathcal{P})$. The following can be found in [23].

Lemma 2. *Let f give an affine covering of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$. Let A be the matrix of the derivative of f. Then both f^A and f_A are translation coverings.*

Proof. This follows by simply writing out the compositions which occur in the definition of an affine covering. □

Remark. *As indicated in Section* 1 *background, there is of course a precise technical definition of the term "Teichmüller disk", again see say* [5]. *More immediate in our setting is the idea of the unit cotangent space of a Teichmüller disk. This is the set of translation surfaces which admit an affine covering of degree 1 to some fixed translation surface. This is, it is the set of surfaces given by the action of $SL(2, \mathbb{R})$ on the atlas of the fixed translation surface. Each translation surface has a natural marked vertical direction, given by the pull-back of the vertical lines of $\mathbb{R}^2$. The Teichmüller disk corresponding to our cotangent space is given by forgetting these marked directions — this indeed allows one to pass from cotangent vector to basepoint in the disk. Note that this is morally equivalent to identifying points in the cotangent space in the same $SO(2, \mathbb{R})$-orbit.*

Remark. *From Lemma* 2, *by a change of surface in either the Teichmüller disk of M or in that of N, we can replace an affine covering $f : M \to N$ by a translation covering. Thus, we may assume that f is a (ramified) covering of Riemann surfaces.*

2.4. Pulling-back 1-forms

Let $f : M \to N$ be a holomorphic map of Riemann surfaces and α a 1-form on N. Then the pull-back, $f^*\alpha$ is a 1-form on M. The following is related to a discussion in [6]. Recall that the ramification points of f are those points of M where the derivative of f vanishes, let us denote these by $Z(f')$. The branch points of f are the images under f of the ramification points, let us denote them by $Br(f)$.

Lemma 3. *Let $f : M \to N$ be a holomorphic map of Riemann surfaces and α a 1-form on N. Let $\mathcal{S}$ be the union of $Br(f)$ and any set of points of N; let $\mathcal{T} = f^{-1}(\mathcal{S})$. If $\mathcal{A} = \mathcal{P}(\alpha; \mathcal{S})$ and $\mathcal{B} = \mathcal{P}(f^*\alpha; \mathcal{T})$, then the map f gives a translation covering of $(N, \mathcal{A})$ by $(M, \mathcal{B})$.*

Proof. To be a translation covering, f must in particular send the marked points of $(M, \mathcal{B})$ to those of $(N, \mathcal{A})$. Now, the zeros of $f^*\alpha$ are $f^{-1}Z(\alpha) \cup Z(f')$. Indeed, an easy calculation shows that if f is locally of the form $t \mapsto t^r =: z$ and α is locally of the form $z^e\,\mathrm{d}z$, then $f^*\alpha$ is of the form $rt^{(e+1)r-1}\,\mathrm{d}t$. Thus, f sends $Z(f^*\alpha)$ to $Z(\alpha) \cup Br(f)$. By definition, f sends $\mathcal{T}$ to $\mathcal{S}$.

As well, the marked points of $(N, \mathcal{A})$ must have as pre-image those of $(M, \mathcal{B})$. Now, $f^{-1}(Z(\alpha))$ is clearly contained in $Z(f^*\alpha)$. Again by definition, the inverse image of $\mathcal{S}$ is $\mathcal{T}$.

Recall that the local coordinates for the atlas induced by α are given by integration of α on N. Those of $f^*\alpha$ are given by a change of variables in the same manner. That is, fixing t_0, not a zero of $f^*\alpha$ on M, local coordinates are given by $\phi(t) = \int_{t_0}^{t} f^*\alpha$. But, then $\phi(t) = \int_{f(t_0)}^{f(t)} \alpha$. Hence, the images of the coordinate functions for the atlas of $f^*\alpha$ equal those of the corresponding coordinate functions of the atlas of α. Hence, f gives a covering of N'' by M'' which preserves translation structure. As we have already shown that the marked points have appropriate images under f and its inverse, f does give a translation covering. □

Remark. *In the above lemma, one can remove the set $f^{-1}(\mathcal{S}) \cap Z(f^*\alpha)$ from $\mathcal{T}$. This is as for any $\mathcal{U}$, the zeros of $f^*\alpha$ are marked points for $\mathcal{P}(f^*\alpha; \mathcal{U})$.*

2.5. Commensurability results

Given a general translation or affine covering of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$, it seems unclear as to exactly how $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ are related. There is, however, some vague knowledge of their relationship. Recall that subgroups of $PSL(2, \mathbb{R})$ are said to be *commensurate* if they share a common subgroup of finite index in each. They are said to be *commensurable* if a finite index subgroup of one conjugates within $PSL(2, \mathbb{R})$ to give a finite index subgroup in the other.

Warning. We follow the definitions of [7] here. It is also common to use the term commensurable to denote what they call commensurate!

Theorem A (Vorobets; Gutkin–Judge). *If there is a translation covering of $(N, \mathcal{Q})$ by $(M, \mathcal{P})$, then $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ are commensurate.*

Corollary A (Gutkin–Judge). *If there is an affine covering of* $(N, \mathcal{Q})$ *by* $(M, \mathcal{P})$, *then* $\Gamma(\mathcal{P})$ *and* $\Gamma(\mathcal{Q})$ *are commensurable*.

In Lemma 1, we stated that marking extra points on a surface gives rise to subgroups of the Veech group of the 1-form corresponding to the structure. Here we give a criterion for when such a structure is no longer commensurable with the original. Recall that a *cylinder* on a translation surface is a maximal connected collection of homotopic closed geodesics which have the same direction (by way of the local coordinates from $\mathbb{R}^2$). The *modulus* of a cylinder is the ratio of its height (length in the fixed direction) to its width.

The fundamental *Veech criterion* states that a direction on a translation surface is preserved by a parabolic element of the Veech group if and only if the cylinders in the direction have their moduli all rationally related (see [21,23] and also [11]). We call such a direction a *parabolic direction*.

We say that a point of a translation surface *splits a cylinder* if there is a direction on the surface for which the flow decomposes into cylinders such that the point is located in the interior of some cylinder. Note that if a point splits a cylinder, then it does so by creating two new (sub)cylinders, each of the same height. We say that a point of a translation surface *irrationally splits a cylinder* if the point splits the cylinder such that the widths of the new subcylinders are irrational multiples of that of the original cylinder.

Lemma 4. *Let* $\mathcal{P}$ *be a given marked translation structure on a surface M and let* $\mathcal{Q}$, *with* $\mathcal{P} \subset \mathcal{Q}$, *be given by marking a point q which irrationally splits a cylinder in a parabolic direction of* $\mathcal{P}$. *Then* $\Gamma(\mathcal{Q})$ *is incommensurate with* $\Gamma(\mathcal{P})$.

Proof. By Lemma 1, $\Gamma(\mathcal{Q})$ has a finite index subgroup which is contained in $\Gamma(\mathcal{P})$. We show that any such subgroup must be of infinite index in $\Gamma(\mathcal{P})$.

Note that the cylinder lies in some parabolic direction for $\mathcal{P}$. Let S be a parabolic element in $\Gamma(\mathcal{P})$ which fixes this direction. It is easily seen that the Veech criterion fails in our direction for $\mathcal{Q}$. Thus no power of S can be contained in $\Gamma(\mathcal{Q})$. But, each finite index subgroup of $\Gamma(\mathcal{P})$ contains some power of S. Therefore, $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ can have no common subgroup of finite index in each. □

3. Polygonal coverings

In the study of properties of Veech groups, one is quickly led to contemplate translation covers which are related to the paving of one polygon by another. We thus loosely use the term polygonal coverings to refer to such translation covers. The fact that one has the explicit algebraic expressions of [3] for the translation surfaces associated to Euclidean triangles allows for a combination of algebraic and geometric techniques to be applied for polygonal coverings which involve only triangles. (In the generic rational polygon setting there is still an algebraic approach, but the so-called accessory parameters in the Schwarz–Christoffel map prevent this from being explicit. On the other hand, see [24] for the use of this map in certain symmetric cases.)

We establish some notation to be used in the remainder of this paper.

Notation. Let $X(p,q,r)$ and $\omega(p,q,r)$ be the Riemann surface and its holomorphic 1-form associated to the billiard flow on the Euclidean triangle $\mathcal{T}(p,q,r)$. Furthermore, let $\Gamma(p,q,r)$ be the Veech group of $\omega(p,q,r)$. Let $\Delta(p,q,r)$ be the Fuchsian triangle group for the angles $\pi/p, \pi/q, \pi/r$ (see [4]).

Veech [21] showed that the following theorem holds

Theorem B (Veech). *For each $n \geq 5$,*

$$\Gamma(1,1,n-2) = \begin{cases} \Delta(2,n,\infty), & \text{odd} n; \\ \Delta(m,\infty,\infty), & n = 2m. \end{cases}$$

The following generalizes an example of [3].

Proposition 1. *Fix $n \geq 5$ and k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$ and $(k,n) = 1$. Let $X_n := X(1,1,n-2)$ and $Y := X(k,k,n-2k)$. Then X_n and Y are biholomorphically equivalent. Furthermore, X_n is the non-singular Riemann surface associated to the equation $y^2 = 1 - x^n$; the pull-back to X_n from Y of $\omega(k,k,n-2k)$ is $cx^{k-1}\,\mathrm{d}x/y$ for a constant $c = c(n,k)$.*

Proof. Recall that Aurell [3] gave the equation $t^n = [s(1-s)]^{n-k}$ for Y; in these coordinates, the 1-form $\omega(k,k,n-2k)$ is simply $\mathrm{d}s/t$. Of course, one so finds a similar equation for X_n, but Veech [21] determined that X_n is of the equation announced.

Fix a choice of an nth root of 4, which we denote by $4^{1/n}$. Let $g : X_n \to Y$ be given by $(x,y) \mapsto ((1-y)/2, (x/4^{1/n})^{n-k}) = (s,t)$. Since $(k,n) = 1$, there exists $r, l \in \mathbb{Z}$ such that $(r+l)n - lk = 1$. Let $f : Y \to X$ be given by $(s,t) \mapsto (4^{1/n}\ t^l\,[s(1-s)]^r,\ 1-2s) = (x,y)$. One checks that f and g are inverses; they give the necessary biholomorphisms.

On X_n, we have $2y\,\mathrm{d}y = -nx^{n-1}\,\mathrm{d}x$, hence $\mathrm{d}y/x^n = (-n/2)\,x^{-1}\,\mathrm{d}x/y$. Thus, $\mathrm{d}((1-y)/2)\,x^{n-k} = cx^{k-1}\,\mathrm{d}x/y$, for an appropriate c. Therefore, the pull-back from Y by g of $\omega(k,k,n-2k) = \mathrm{d}s/t$ is of the stated form. □

Remark. *The translation surfaces defined by some* 1*-form ω and its multiple by a non-zero complex constant c are virtually the same. The action of $SL(2,\mathbb{R})$ on translation surfaces by way of charts clearly preserves area. Scaling by a real non-zero constant to achieve a surface of area 1 changes no intrinsic aspect of the surface — in particular, Veech groups are preserved under this scaling.*

As well, rotation by $\arg(c)$ (*of the charts of an atlas*) *imposes no intrinsic change — one merely gives a different choice of standard* (*vertical*) *direction. In particular, conjugacy classes of Veech groups are preserved under such rotation. In fact, such rotation preserves the point in Teichmüller space corresponding to the* 1*-form, cotangent vectors are rotated.*

Thus, we will actually work in the projective space of 1*-forms, identifying all non-zero complex multiples of a* 1*-form.*

Remark. *In fact, there is no canonical vertical direction in the construction of* [10]. *The translation surface constructed allows one to follow the billiard flow in any direction (which does not encounter a singularity). That is, dividing by the action of the rotation group* $SO(2, \mathbb{R})$ *is completely natural here.*

Earle and Gardiner [5] show, in our notation, that $\Gamma(2, 2, 1) = \Delta(5, \infty, \infty)$. Indeed, by inspection of their examples, they actually show the following theorem.

Theorem C (Earle–Gardiner). *Let the integer* $k \geq 2$. *Then*

$$\Gamma(2k-1, 2k-1, 2) = \Delta(2k, \infty, \infty), \qquad \Gamma(k, k, 1) = \Delta(2k+1, \infty, \infty).$$

Thus, for odd $n = 2k+1$ the 1-form $\omega(1, 1, n-2)$ and $\omega(k, k, 1)$ on X_n both give lattices. Earle and Gardiner [5] remark that the case of $n = 5$ gives two linearly independent 1-form on X_5 which have lattices for Veech groups and note that this is an interesting phenomenon. Harvey [8,9] points out that this phenomenon may exist for other surfaces. Here we show that in fact there is no universal bound on the number of 1-forms on a Riemann surface which have incommensurable lattice Veech groups.

We first determine the Veech group of a certain family of marked translation structures.

Proposition 2. *Let* $j \in \mathbb{N}$, $j \geq 2$ *and* $n = 2j + 1$. *Let* X_n *be the non-singular Riemann surface of affine equation* $y^2 = 1 - x^n$ *and let* ∞ *be the unique point at infinity on* X_n. *Then* $\Gamma(x^{j-1}\, \mathrm{d}x/y;\ \infty) = \Gamma(j, j, 1)$.

Proof. It is easily calculated that the genus of X_n is j, and that all of the zeros of $x^{j-1}\, \mathrm{d}x/y$ on that surface occur at $(0, \pm 1)$.

In order to establish our result, we wish to locate the zeros and ∞ on a geometric realization of X_n. But, X_n is $X(j, j, 1)$, the Riemann surface associated to $\mathcal{T}(j, j, 1)$; furthermore, by Proposition 1, $\omega(j, j, 1)$ is the 1-form (up to a negligible constant) $x^{j-1}\, \mathrm{d}x/y$ on X_n. Earle and Gardiner's proof of Theorem C begins with the construction of what one recognizes as the surface $X(j, j, 1)$. This is a $2n$-gon with opposite faces identified; it can be tiled by $2n$ copies of $\mathcal{T}(j, j, 1)$. Zeros for the associated 1-form are found at the external vertices of this regular figure.

We need to now locate the point ∞ on the $2n$-gon. To this end, we turn to the Aurell–Itzykson approach. The Aurell–Itzykson equation for this X_n is $t^n = [s(1-s)]^{j+1}$. The appropriate map g of the proof of Proposition 1 sends $(x, y) = (0, \pm 1)$ to the points where $s = 0$ and $s = 1$. We continue to call the single point at infinity in both coordinate systems simply ∞.

Aurell and Itzykson [3] determined the equation for X_n (indeed, for any surface associated to a rational triangular billiard table) by using the Schwarz triangle function: the upper half-plane is mapped to the interior of the triangle. Normalization is such that 0, 1 and ∞ (on the Riemann sphere) are sent to the vertices of the triangle. Schwarz reflection allows one to extend the inverse of the triangle map to all of the associated surface. (In particular, by the famed result of Belyi, every such surface has an equation defined over $\bar{\mathbb{Q}}$ — see [25].)

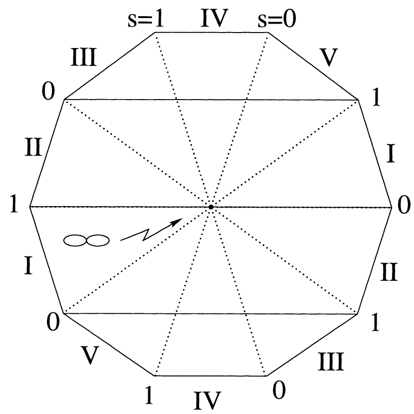

Fig. 1. Case of $j = 2 : X(2, 2, 1)$; horizontal cylinders marked.

The Aurell–Itzykson determination of the equation implies that the vertices of the tiling of X_n by $\mathcal{T}(j, j, 1)$ occur where $s \in \{0, 1, \infty\}$. But, we already know that our zeros occur where $s \in \{0, 1\}$ and lie at the external vertices of the regular figure of Earle and Gardiner. Hence, the point ∞ is at the center (see Fig. 1).

We now show that the Veech group $\Gamma(\omega(j, j, 1); \infty)$ equals $\Gamma(j, j, 1)$. Following Veech, Earle and Gardiner identify a generating pair of elements for $\Gamma(j, j, 1)$: the central rotation and a parabolic element in the horizontal direction. The first of these clearly fixes the center of the $2n$-gon. The center lies on the boundary of a cylinder for the parabolic element. One easily checks that this element fixes each point of this boundary. Thus all of $\Gamma(j, j, 1)$ fixes the center. (Hence, the Veech group of the surface punctured at the center is exactly the same as that of the unpunctured surface.) Since this center is indeed the point ∞, we have our equality of Veech groups. □

Remark. *Earle and Gardiner* [5] *also treat the case of 4n-gons with opposite sides identified. For further treatment of these surfaces, see* [2].

We pull-back various of the marked structures of Proposition 2 to find Riemann surfaces which have numerous 1-forms with non-trivial Veech groups.

Theorem 1. *Let $L \in \mathbb{N}$, then there exists a Riemann surface with 1-forms $\omega_1, \dots, \omega_L$, such that the $\Gamma(\omega_i)$ are pairwise incommensurable lattices.*

Proof. Let m be the product of the first L odd numbers, starting with 5. Thus, let $m(j) = 2j + 1$ and $m = \prod_{j=2}^{L+1} m(j)$. We use Veech's equations for the various X_n: $y^2 = 1 - x^n$. For odd n, the non-singular Riemann surface X_n has a single point at infinity (with respect to these coordinates). There are maps $f_j : X_m \to X_{m(j)}$ given by $(x, y) \mapsto (x^{m/m(j)}, y)$. Note that f_j has its branch points at $(0, \pm 1)$ and at the single point of $X_{m(j)}$ at infinity (which we simply denote by ∞). Note that X_m also has a single point at infinity, this is the sole pre-image of ∞ under f_j.

Let ω_j be the pull-back by f_j of $x^{j-1}\,\mathrm{d}x/y$ on $X_{m(j)}$. As $Z(x^{j-1}\,\mathrm{d}x/y) = (0, \pm 1)$, Lemma 3 and Theorem A show that $\Gamma(\omega_j)$ is commensurate with $\Gamma(x^{j-1}\,\mathrm{d}x/y;\, \infty)$. By Proposition 2, this is $\Gamma(j, j, 1)$. Thus, by Theorem C, $\Gamma(\omega_j)$ is commensurable with $\Delta(m(j), \infty, \infty)$. But, the group $\Delta(m(j), \infty, \infty)$ is a subgroup of index 2 in the triangle group $\Delta(2, 2m(j), \infty)$. Such triangle groups are lattices; since this property is shared by finite index subgroups, the $\Gamma(\omega_j)$ are indeed lattices.

Triangle groups are unique up to $PSL(2, \mathbb{R})$ conjugation. Any $\Delta(2, k, \infty)$ is hence conjugate to the so-called Hecke group of index k, see say [4]. Now, Leutbecher [14] showed that (except for $k \in \{3, 4, 6\}$) these are all pairwise incommensurable. That is, the $\Gamma(\omega_j)$ are also pairwise incommensurable. □

Remark. *The aforementioned result of Leutbecher has since been greatly generalized by Margulis* [16], *see also* [15]. *Briefly, the commensurability class of a non-arithmetic triangle Fuchsian group possesses a unique maximal element* (*up to conjugation*).

We have just used pull-backs of 1-forms to find many 1-forms on a single Riemann surface which have lattice Veech groups. We will soon pull-back 1-forms such that the original 1-form has a lattice Veech group, but the pulled-back 1-form does not. We use the following proposition in this construction. Note that X_n continues to denote the non-singular Riemann surface associated to the equation $y^2 = 1 - x^n$.

Proposition 3. *Let $n \geq 5$ be an odd integer and let $\mathcal{P}_n = \mathcal{P}(\omega(1, 1, n-2);\, p_1, p_2)$, where the p_i are the points $(x, y) = (0, \pm 1)$ on X_n. Then, the Veech group $\Gamma(\mathcal{P}_n)$ is not a lattice.*

Proof. We apply Lemma 4. As Theorem B states, Veech [21] showed that $\Gamma(1, 1, n-2)$ is a lattice. We show that $\Gamma(\mathcal{P}_n)$ is of infinite index, and thus cannot be a lattice.

In fact, we also return to calculations of Veech [21]. Veech constructs the surfaces X_n by taking two copies of the regular n-gon and gluing them appropriately. While finding his equation for X_n, he shows (by a use of the symmetry group of the surface) that the centers of these correspond to the points of coordinates $(0, \pm 1)$ [21, Section 4.4].

By the symmetry of his construction, we may choose one of the n-gons, with its vertices at the nth roots of unity. Consider the vertical foliation and its cylinders. Veech [21, Section 5] shows that there are cylinders whose vertical boundaries pass through the various $x_j = \cos 2\pi j/n$. Fix that j is such that $x_{j+1} < 0 < x_j$. It is easily seen that $j = \lfloor n/4 \rfloor$ (see Fig. 2).

The marked point of our n-gon splits the cylinder whose boundaries pass through x_j and x_{j+1}. By Lemma 4, it suffices to show that x_j is not a rational multiple of $x_j - x_{j+1}$. Of course, this is the same as showing that x_j and x_{j+1} are rationally independent. Up to the same constant factor which we simply suppress, these are $\zeta^j + \zeta^{-j}$ and $\zeta^{j+1} + \zeta^{-j-1}$, where $\zeta = \mathrm{e}^{2\pi \mathrm{i}/n}$.

Were $(\zeta^{j+1}+\zeta^{-j-1})/(\zeta^j+\zeta^{-j})$ rational, then it would be invariant under every element of the Galois group of the field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$. For all n we are considering, we have the non-trivial element $\sigma : \zeta \mapsto \zeta^2$. Thus, we will show that $(\zeta^{2j+2}+\zeta^{-2j-2})/(\zeta^{2j}+\zeta^{-2j}) = (\zeta^{j+1} + \zeta^{-j-1})/(\zeta^j + \zeta^{-j})$ is impossible for our value of j. We clear denominators,

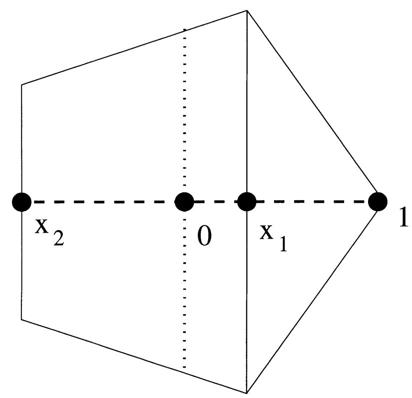

Fig. 2. Case of $n = 5$: a single regular pentagon, with vertical cylinders and center marked.

and have $\zeta^{3j+1} + \zeta^{-3j-1} + \zeta^{j-1} + \zeta^{-j+1}$ being equal to $\zeta^{3j+2} + \zeta^{-3j-2} + \zeta^{j+2} + \zeta^{-j-2}$.

If $n = 4j + 1$, then $3j + 2 \equiv -j + 1 \bmod n$ and similarly for $3j + 1$. Hence if $\zeta^{3j+1} + \zeta^{-3j-1} + \zeta^{j-1} + \zeta^{-j+1}$ is equal to $\zeta^{3j+2} + \zeta^{-3j-2} + \zeta^{j+2} + \zeta^{-j-2}$, then $\zeta^{j} + \zeta^{-j} = \zeta^{j+2} + \zeta^{-j-2}$. However, by our choice of j, the first of these is negative and the second positive. The analogous argument leads to a similar contradiction when $n = 4j + 3$. □

Corollary. *Let k, l and m be natural numbers, such that $k \geq 2$, $l \geq 5$ is odd and $m = kl$. If $\alpha = x^{k-1}\,\mathrm{d}x/y$ on the surface X_m, then $\Gamma(\alpha)$ is not a lattice.*

Proof. Let $f : X_m \to X_l$ be given by $(x, y) \to (x^k, y)$. Then f is branched at the points of coordinates $(0, \pm 1)$ and (depending upon k) possibly at the single point at infinity. Let $\omega = \omega(1, 1, l-2)$; recall that ω has its sole zero at the point at infinity. By Lemma 3, $f : (X_m, f^*\omega) \to (X_l, \mathcal{P}_l)$ is a translation covering. Therefore, by Theorem A, $\Gamma(f^*\omega)$ is commensurable with $\Gamma(\mathcal{P}_l)$. But, up to a constant, $f^*\omega = x^{k-1}\,\mathrm{d}x/y$. Thus, the proposition completes the proof. □

Let us use the term *Veech group of a (rational angled) polygon* to denote the Veech group of the translation surface defined in the standard [10] manner. We now give an infinite family of triangles whose Veech groups are not lattices.

Proposition 4. *Let $n \geq 5$ be an odd natural number. Then $\Gamma(n-2, n-2, 4)$ is not a lattice.*

Proof. We give two proofs. Although closely related, these emphasize different aspects of the surfaces involved.

Algebraic proof. By Proposition 1, $\omega(n-2, n-2, 4)$ is $x^{n-3}\,\mathrm{d}x/y$ on X_{2n}. Consider the map h from X_{2n} to itself which maps (x, y) to $(1/x, \mathrm{i}y/x^n)$. Let $\alpha = x\,\mathrm{d}x/y$ on X_{2n}.

Then, pull-back by h of α is (up to constants)

$$h^*\alpha = h^*x\,\mathrm{d}x/y = x^{-1}(x^{-2}\,\mathrm{d}x)/x^{-n}y = x^{n-3}\,\mathrm{d}x/y.$$

Since h is injective, it is clearly unramified. Therefore $\Gamma(n-2, n-2, 4) = \Gamma(\alpha)$. By the above corollary (with $k = 2$ and $l = n$), $\Gamma(\alpha)$ is not a lattice. Therefore our proposition is proved.

Constructive proof. We begin with the triangle $\mathcal{T}(n-2, n-2, 4)$. The corresponding translation surface, $M := X(n-2, n-2, 4)$, will be tiled by $4n$ copies of this triangle. We can begin by developing about the vertex of angle $2\pi/n$. That is, we consecutively flip copies of $\mathcal{T}(n-2, n-2, 4)$ about this vertex. Consecutive copies under these flips have opposite orientations. Since n is odd, after completing an angle of 2π, the next flip gives a copy which has the opposite orientation from that of the initial copy. Thus this ensuing copy cannot be identified with the initial copy of the triangle. In fact, one needs an angle of 4π to complete this conical singular point on M. First, we may take n more copies of the triangle and place them along the edges of these first n so as to make a regular stellated n-gon. To complete the development about the conical point, we take a second stellated regular n-gon and glue the two along a slit running (straight) from the center to an exterior vertex of the stellated n-gon (see Fig. 3).

The single point represented by the centers of these two stellated regular n-gons is a conical singularity on M of angle 4π; the point represented by the exterior vertices is also singular of angle 4π. There are two other singular points, each of angle $(n \times 4(n-2)\pi/2n =)$ $(n-2)2\pi$.

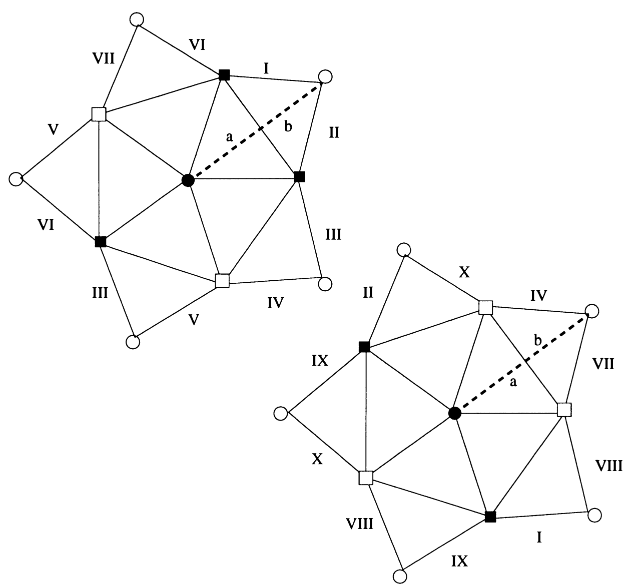

Fig. 3. $X(3, 3, 4)$ tiled by $\mathcal{T}(3, 3, 4)$.

Now, we consider the triangle $\mathcal{T}(1, 1, n-2)$. Let $N := X(1, 1, n-2)$. Then we can obtain N by developing $2n$ copies of the triangle about one of the vertices of angle π/n. The regular stellar figure for N so given is, up to identifications on its exterior edges, exactly of the form of each of those found for M.

Thus, in terms of Riemann surfaces, M is a double cover of N, which is branched at the points corresponding to the center and to the exterior vertices. We can either argue with the Aurell–Itzykson determination of equations (as in the proof of Proposition 2), or take Veech's [21] figure for N and cut, translate and paste to find that these branch points are indeed $(x, y) = (0, \pm 1)$. Thus, by Lemma 3, we have that $\Gamma(n-2, n-2, 4) = \Gamma(\mathcal{P}_n)$. Therefore, by Proposition 3, $\Gamma(n-2, n-2, 4)$ is not a lattice. □

The above corollary and proposition lead to an infinite family of counter-examples to the naive intuition that if a polygon tiles another (where one admits both translations and *flips* across a boundary edge as tiling "moves"), then the Veech groups of these polygons are commensurable. We thank J. Smillie for pointing out that a remark in passing in [23] already mentions this aspect of our polygons.

Theorem 2 (Vorobets). *There exist polygons P and Q such that P tiles Q by flips and the Veech groups of the corresponding translation surfaces are not commensurable.*

Proof. Fix an odd integer $n \geq 5$. Consider the right triangle $\mathcal{T}(2, n-2, n)$. The associated surface is formed by $4n$ copies of this triangle. If we flip the triangle about its right angle, we find $\mathcal{T}(4, n-2, n-2)$ and $\mathcal{T}(1, 1, n-2)$ (see Fig. 4).

As explained in the constructive proof of Proposition 4, the surface of $X(1, 1, n-2)$ is tiled by $2n$ copies of $\mathcal{T}(1, 1, n-2)$. Indeed, again because n is odd, one easily verifies that the surface for this and for the right triangle are translation isomorphic — they can be formed by taking the same region of $\mathbb{R}^2$ and identifying sides in exactly the same way. By Theorem B, the Veech group $\Gamma(1, 1, n-2)$ is a lattice; it follows that the Veech group $\Gamma(2, n-2, n)$ is also a lattice. But, Proposition 4 shows that the Veech group of $\mathcal{T}(4, n-2, n-2)$ is not

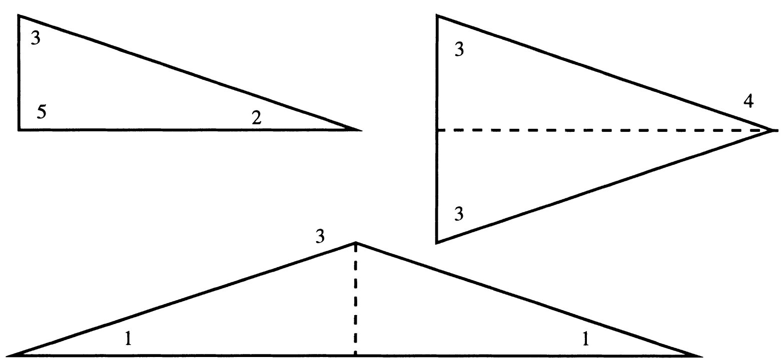

Fig. 4. Case of $n = 5$: $\mathcal{T}(2, 3, 5)$, $\mathcal{T}(4, 3, 3)$ and $\mathcal{T}(1, 1, 3)$.

a lattice. Thus, although the triangle $\mathcal{T}(2, n-2, n)$ tiles $\mathcal{T}(4, n-2, n-2)$, we find that their Veech groups are not commensurable. □

There are similar tilings of triangles by triangles where Veech groups are preserved, as we now show.

Example. Let $n = 2m$ be an even integer, $n \geq 6$. Consider the right triangle $\mathcal{T}(2, n-2, n)$. Just as in the above proof, we have the two related triangles $\mathcal{T}(2, 2, 2(n-2)) = \mathcal{T}(1, 1, n-2)$ and $\mathcal{T}(4, n-2, n-2) = \mathcal{T}(2, m-1, m-1)$. By Theorem B, the first of these has Veech group $\Delta(m, \infty, \infty)$. By Theorem C, the second has the same Veech group. Indeed, it is easily checked that the surface of this second triangle is isomorphic to the surface of the right triangle.

This cutting, flipping and then translating, as in the passage from $\mathcal{T}(1, 1, n-2)$ to $\mathcal{T}(2, m-1, m-1)$ above, is an integral step in classification of those acute triangles which have lattice Veech groups [11].

We now show that Veech groups usually are preserved up to commensurability under polygonal tiling by flips.

Proposition 5. *If a rational polygon has no angles of the form π/n for integer n, then any polygon which it tiles by a single flip has a commensurable Veech group. Furthermore, if a rational polygon has no angles of the form $2\pi/n$ for integer n, then any polygon which it tiles by flips has a commensurable Veech group.*

Proof. Let Q be a polygon tiled by the polygon P and let $M(Q)$ and $M(P)$ be their associated surfaces. That there is a covering of the Riemann surfaces determined by $M(Q)$ and $M(P)$ can be seen for instance by use of Proposition 5.1 of [23]. There it is already shown that the sole obstruction to the covering of $M(P)$ by $M(Q)$ being a translation cover is given by the restriction that the covering map and its local inverse send the sets of singularities to one another.

The singularities of a translation surface of a polygon can only occur at points which project to vertices of the polygon. We thus will discuss the set of such points on each of $M(Q)$ and $M(P)$.

Suppose that P has no angles of the form π/n. Then every point of $M(P)$ which projects to a vertex of P is singular. Since the vertices of Q lie at vertices of paving copies of P, the covering map certainly sends the singularities of $M(Q)$ to those of $M(P)$.

Continuing with the assumption that P has no angles of the form π/n, none of the vertices of Q can be of the form π/n for integer n. This follows by simply writing each angle of P in the form $l\pi/m$ with l, m relatively prime and $2 \leq l < 2m$. A vertex of Q has a multiple of such an angle; were this multiple to be $1/n$, then l, m would not be relatively prime. Thus, also every point of $M(Q)$ which projects to a vertex of Q is singular.

It now suffices to show that the inverse images of the singularities of $M(P)$ are the singularities of $M(Q)$. Since every point of $M(P)$ which projects to a vertex of P is singular, and similarly for $M(Q)$, we need only show that the vertices of the paving copies of P lie at the vertices of Q.

1. If Q is the union of P and a single flip of P, then each vertex of these two copies of P on Q lies at some vertex of Q.
2. Suppose now that P has no angles of the form $2\pi/n$. Vertices of paving copies of P meet at points of Q of angles which are integer multiples of the angles of P; these multiples are never equal to 2π, hence vertices of paving copies of P lie at interior points of Q. Our argument showing that Q has no angles of the form π/n shows that no paving copies of P can meet at the interior of an edge of Q. Hence, the vertices of paving copies of P lie at the vertices of Q. □

Remark. *Gutkin and Judge* [6,7] *have characterized those translation surfaces whose Veech groups are arithmetic (i.e., commensurable to* $PSL(2,\mathbb{Z})$*): these are the surfaces which can be tiled by translations of Euclidean parallelograms*.

Acknowledgements

We thank Pierre Arnoux for helpful discussions, C. Judge for an elucidating e-mail message, and J. Smillie for comments on an earlier version. We thank E. Gutkin for numerous remarks on this work. We also thank the referee for suggestions to improve the presentation. The second author thanks the Max Planck Institut and the Institut de Mathématiques de Luminy for providing stimulating atmospheres.

References

[1] P. Arnoux, Ergodicité générique des billards polygonaux, (d'aprés Kerkhoff, Masur, Smillie), Séminaire Bourbaki 1987/1988 (1989) 203–221; Astérisque No. 161–162 (1988) No. 696, 5.

[2] P. Arnoux, P. Hubert, Fractions continues sur les surfaces de Veech, J. d'Analyse Math., in press.

[3] E. Aurell, C. Itzykson, Rational billiards and algebraic curves J. Geom. Phys. 5 (1988) 191–208.

[4] A. Beardon, The Geometry of Discrete Groups, Grad. Text Math., Vol. 91, Springer, Berlin, 1984.

[5] C.J. Earle, F.P. Gardiner, Teichmüller disks and Veech's $\mathcal{F}$-structures, in: J.R. Quine, P. Sarnak (Eds.), Extremal Riemann Surfaces, Contemp. Math., Vol. 201, American Mathematical Society, Providence, RI, 1997, pp. 165–189.

[6] E. Gutkin, C. Judge, The geometry and arithmetic of translation surfaces with applications to polygonal billiards Math. Res. Lett. 3 (1996) 391–403.

[7] E. Gutkin, C. Judge, Affine mappings of translation surfaces: geometry and arithmetic, Duke Math. J., in press.

[8] W. Harvey, On certain families of compact Riemann surfaces, in: J.R. Quine, P. Sarnak (Eds.), Mapping Class Groups and Moduli Spaces of Riemann Surfaces, Contemp. Math., Vol. 150, American Mathematical Society, Providence, RI, 1993, pp. 137–148.

[9] W. Harvey, Drawings, triangle groups and algebraic curves, Preprint, 1997.

[10] A. Katok, A. Zemlyakov, Topological transitivity of billiard flows in polygons Math Notes 18 (1975) 760–764.

[11] R. Kenyon, J. Smillie, Billiards in rational-angled triangles, Commentarii Math Helvetici, in press.

[12] S.P. Kerkhoff, H. Masur, J. Smillie, Ergodicity of billiard flows and quadratic differentials Ann. Math. 124 (1986) 293–311.

[13] I. Kra, The Carathéodory metric on abelian Teichmüller disks J. d'Analyse Math. 40 (1981) 129–143.

[14] A. Leutbecher, Über die Heckeschen Gruppen $G(\lambda)$, II Math. Ann. 211 (1974) 63–86.
[15] C. Maclachlan, G. Rosenberger, Commensurability Classes of Two Generator Fuchsian Groups, Discrete Groups and Geometry, Lecture Note Series 173, London Mathematical Society, Cambridge University Press, Cambridge, 1992, pp. 171–189.
[16] G.A. Margulis, Discrete Subgroups of Semisimple Lie Groups, Springer, New York, 1991.
[17] H. Masur, Closed geodesics for quadratic differentials with applications to billiards Duke J. Math. 53 (1986) 307–314.
[18] H. Masur, Z. Tabachnikov, Rational billiards and flat structures, Vol. 55, Max Planck Institut für Mathematik, Bonn, Preprint, 1999.
[19] Z. Tabachnikoff, Billiards Panoramas et Synthèses, Vol. 1, Soc. Math. France, Paris, 1995.
[20] M. Troyanov, Les surfaces euclidiennes à singularités coniques Enseign. Math. 32 (2) (1986) 79–94.
[21] W. Veech, Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards Invent. Math. 97 (1989) 553–583.
[22] W. Veech, The billiard in a regular polygon Geom. Funct. Anal. 2 (1992) 341–379 .
[23] Ya. B. Vorobets, Plane structures and billiards in rational polyhedra: the Veech alternative (Russian), Uspekhi Mat. Nauk 51 (1996) 3–42 [Russian Math Surveys 51 (5) (1996) 779–817 (translation)].
[24] C. Ward, Calculation of Fuchsian groups associated to billiards in a rational triangle Ergodic Theory Dynam. Systems 18 (1998) 1019–1042.
[25] J. Wolfart, The "obvious" part of Belyi's theorem and Riemann surfaces with many automorphisms, in: L. Schneps (Ed.), Geometric Galois Actions, Lecture Note Series 242, London Mathematical Society, Cambridge University Press, Cambridge, 1997, pp. 97–112.